5-3 The Pythagorean Theorem and Its Converse

What You'll Learn

- Using the Pythagorean Theorem and its converse

...And Why

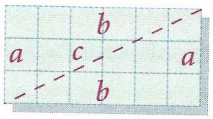
To solve problems involving boundaries, packaging, and satellites

What You'll Need

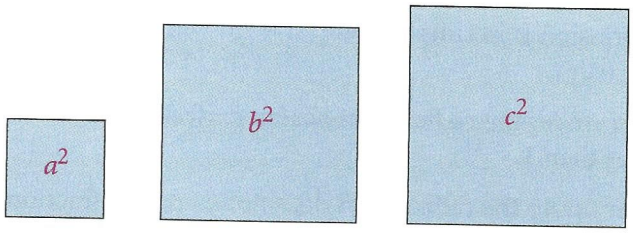
scissors, graph paper, colored paper, straightedge

WORK TOGETHER

Work in groups. Using graph paper, draw any rectangle. Label the sides a and b. Cut four rectangles with length a and width b from the graph paper. Then cut each rectangle on its diagonal, c, forming eight congruent triangles.



Cut three squares from the colored paper, one with sides of length a, one with sides of length b, and one with sides of length c.



Separate the pieces into groups.

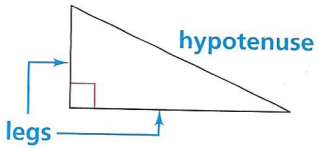
Group 1: four triangles and the two smaller squares

Group 2: four triangles and the largest square

Arrange the pieces of each group to form a square.

1. Write an algebraic expression for the area of each of the squares you formed.
2. How do the areas of the two squares you formed compare?
3. What can you conclude about the areas of the squares you cut from colored paper?
4. Express your conclusion as an algebraic equation.

THINK AND DISCUSS



The Pythagorean Theorem

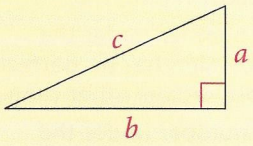
In a right triangle, the side opposite the right angle is the longest side. It is the **hypotenuse.** The other two sides are the **legs of a right triangle.**

The Work Together presents a justification of the well-known right triangle relationship called the Pythagorean Theorem.

Theorem 5-4
Pythagorean Theorem

In a right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.

$$a^2 + b^2 = c^2$$

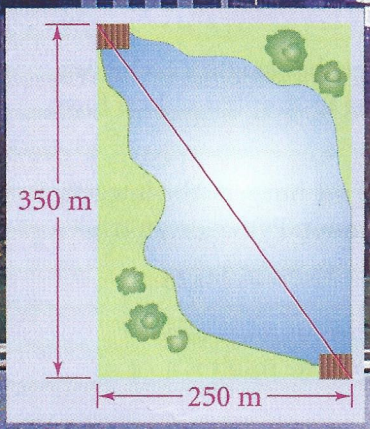


5. a. A right triangle has sides of lengths 20, 29, and 21. What is the length of the hypotenuse?

b. **Verify** that the Pythagorean Theorem is true for the right triangle in part (a).

Example 1 **Relating to the Real World**

Recreation A city park department rents paddle boats at docks near each entrance to the park. About how far is it to paddle from one dock to the other?



You can find the distance between the two docks by finding the hypotenuse of the right triangle.

$$a^2 + b^2 = c^2$$ Use the Pythagorean Theorem.

$$250^2 + 350^2 = c^2$$ Substitute 250 for a and 350 for b.

$$62{,}500 + 122{,}500 = c^2$$ Simplify.

$$185{,}000 = c^2$$

$$c = \sqrt{185{,}000}$$ Find the square root.

185,000 √ = 430.11626

It is about 430 m from one dock to the other.

6. Try This Find the length of the hypotenuse of a right triangle with legs of lengths 7 and 24.

QUICK REVIEW

A radical expression is in simplest radical form when all the following are true.

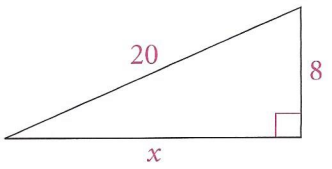
- The number under the radical sign has no perfect square factors other than 1.
- The number under the radical sign does not contain a fraction.
- The denominator does not contain a radical expression.

For practice with radical expressions, see Skills Handbook page 660.

Sometimes you will leave your answer in simplest radical form.

Example 2

Find the value of x. Leave your answer in simplest radical form.



Use the Pythagorean Theorem.

$a^2 + b^2 = c^2$

$8^2 + x^2 = 20^2$ **Substitute.**

$64 + x^2 = 400$ **Simplify.**

$x^2 = 336$ **Subtract 64 from each side.**

$x = \sqrt{336}$ **Find the square root.**

$x = \sqrt{16(21)}$ **Simplify.**

$x = 4\sqrt{21}$

7. Try This The hypotenuse of a right triangle has length 12. One leg has length 6. Find the length of the other leg in simplest radical form.

When the lengths of the sides of a right triangle are integers, the integers form a **Pythagorean triple.** Here are some common Pythagorean triples.

3, 4, 5 5, 12, 13 8, 15, 17 7, 24, 25

8. Open-ended Choose an integer. Multiply each number of a Pythagorean triple by that integer. **Verify** that the result is a Pythagorean triple.

The Converse of the Pythagorean Theorem

You can use the Converse of the Pythagorean Theorem to determine whether a triangle is a right triangle.

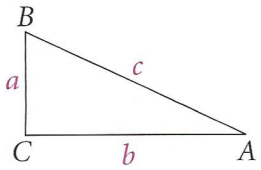
Theorem 5-5
Converse of the Pythagorean Theorem

If the square of the length of one side of a triangle is equal to the sum of the squares of the lengths of the other two sides, then the triangle is a right triangle.

Who? Czech-American mathematician Olga Taussky-Todd (1906–1995) studied Pythagorean triangles. In 1970, she won the Ford Prize for her research.

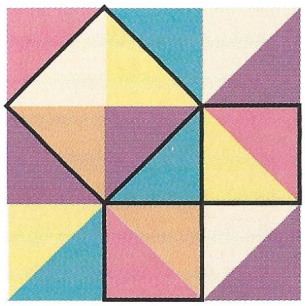
The Converse of the Pythagorean Theorem leads to the inequalities below. You can use them to determine whether a triangle is obtuse or acute.

In $\triangle ABC$ with longest side c,
if $c^2 > a^2 + b^2$, then the triangle is obtuse, and
if $c^2 < a^2 + b^2$, then the triangle is acute.



When? The Pythagorean Theorem is named for Pythagoras, a Greek mathematician who lived in the sixth century B.C. We now know that the Babylonians, Egyptians, and Chinese were aware of this relationship before Pythagoras.

The diagram below illustrates an ancient Greek proof of the Pythagorean Theorem for an isosceles right triangle.



Example 3

The numbers represent the lengths of the sides of a triangle. Classify each triangle as acute, obtuse, or right.

a. 13, 84, 85

$85^2 \stackrel{?}{=} 13^2 + 84^2$ — Compare c^2 to $a^2 + b^2$. Substitute the length of the longest side for c.

$7225 \stackrel{?}{=} 169 + 7056$

$7225 = 7225$ — $c^2 = a^2 + b^2$

The triangle is a right triangle.

b. 6, 11, 14

$14^2 \stackrel{?}{=} 6^2 + 11^2$ — Compare c^2 to $a^2 + b^2$. Substitute the length of the longest side for c.

$196 \stackrel{?}{=} 36 + 121$

$196 > 157$ — $c^2 > a^2 + b^2$

The triangle is an obtuse triangle.

9. Try This A triangle has sides of lengths 7, 8, and 9. Classify the triangle as acute, obtuse, or right.

Exercises ON YOUR OWN

Algebra **Find the value of x. Leave your answer in simplest radical form.**

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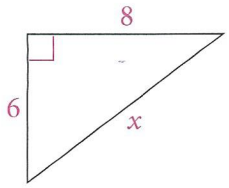

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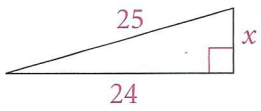

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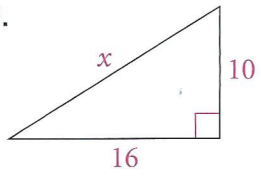

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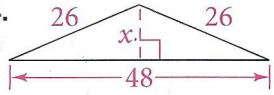

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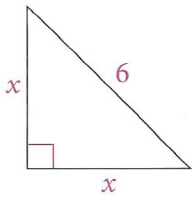

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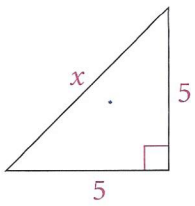

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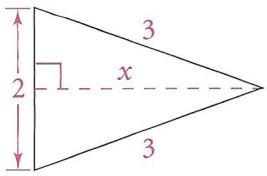

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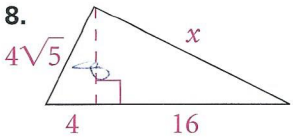


9. A 15-ft ladder is leaning against a building. The base of the ladder is 5 ft from the building. To the nearest foot, how high up the building does the ladder reach?

PROBLEM SOLVING HINT
Draw a diagram.

10. A brick walkway forms the diagonal of a square playground. The walkway is 24 m long. To the nearest tenth of a meter, how long is a side of the playground?

Choose **Use mental math, paper and pencil, or a calculator. The lengths of the sides of a triangle are given. Classify each triangle as acute, right, or obtuse.**

11. 15, 8, 21

12. 12, 16, 20

13. $2, 2\frac{1}{2}, 3$

14. 30, 34, 16

15. 0.3, 0.4, 0.6

16. 11, 12, 15

17. $\sqrt{3}, 2, 3$

18. 1.8, 8, 8.2

19. 20, 21, 28

20. 31, 23, 12

21. 30, 40, 50

22. $\sqrt{11}, \sqrt{7}, 4$

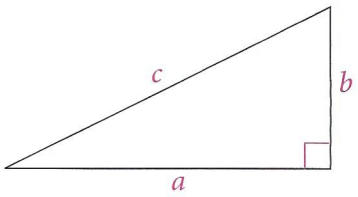
23. **Ancient Egypt** Each year the Nile River overflowed its banks and deposited fertile silt on the valley farmlands. Although the flood was helpful to farmers, it often destroyed boundary markers. Egyptian surveyors used a rope with knots at 12 equal intervals to help reconstruct boundaries.

a. **Writing** Explain how a surveyor could use this rope to form a right angle.

b. **Research** Find out why the Nile no longer floods as it did in ancient Egypt.

24. **Open-ended** Draw a right triangle with three sides that are integers. Draw the altitude to the hypotenuse. Label the lengths of the three sides and the altitude.

Calculator **Use the triangle at the right. Find the missing length to the nearest tenth.**



25. $a = 3, b = 7, c =$ ■

26. $a = 1.2, b =$ ■, $c = 3.5$

27. $a =$ ■, $b = 23, c = 30$

28. $a = 0.7, b =$ ■, $c = 0.8$

29. $a = 8, b = 8, c =$ ■

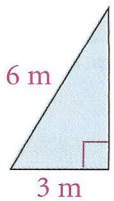
30. $a =$ ■, $b = 9, c = 18$

31. **Sewing** You want to embroider a square design. You have an embroidery hoop with a 6-in. diameter. Find the largest value of x such that the entire square will fit in the hoop. Round to the nearest tenth.

32. A rectangle has 10-in. diagonals and the lengths of its sides are whole numbers. Use the problem-solving strategy *Guess and Test* to find the perimeter of the rectangle.

Find the area of each figure. Leave your answer in simplest radical form.

33.



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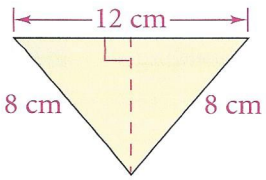

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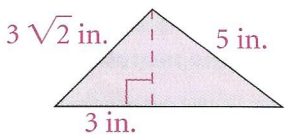

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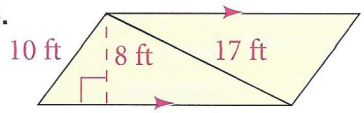
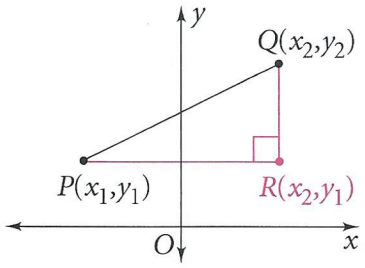


37. Coordinate Geometry You can use the Pythagorean Theorem to prove the Distance Formula. Let points $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the endpoints of the hypotenuse of a right triangle.



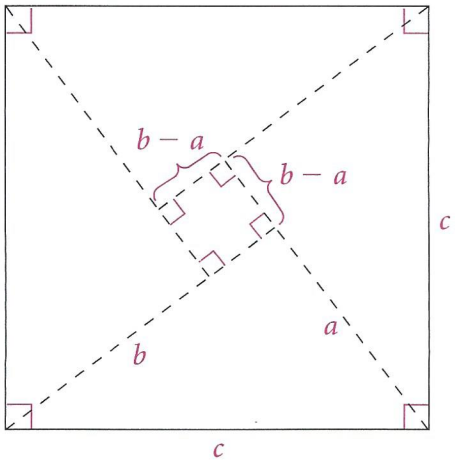
- **a.** Write an algebraic expression to complete each of the following: $PR =$ ■ and $QR =$ ■.
- **b.** By the Pythagorean Theorem, $PQ^2 = PR^2 + QR^2$. Rewrite this statement, substituting the algebraic expressions you found for PR and QR in part (a).
- **c.** Complete the proof by finding the square root of each side of the equation that you wrote in part (b).

Find a third number so that the three numbers form a Pythagorean triple.

38. 9, 41 **39.** 14, 48 **40.** 60, 61 **41.** 8, 17

42. 20, 21 **43.** 13, 85 **44.** 12, 37 **45.** 63, 65

46. Logical Reasoning You can use the diagram at the right to prove the Pythagorean Theorem.



- **a.** Find the area of the large square in terms of c.
- **b.** Find the area of the large square in terms of a and b by finding the area of the four triangles and the small square.
- **c.** Write an equation setting your answers to part (a) and part (b) equal to each other. Complete the proof by simplifying the equation.

Calculator The figures below are drawn on centimeter graph paper. Find the perimeter of each shaded figure to the nearest tenth.

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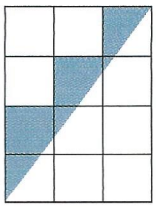

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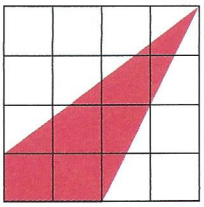

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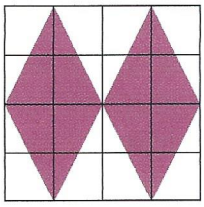

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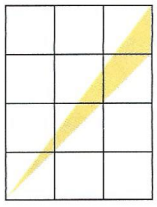
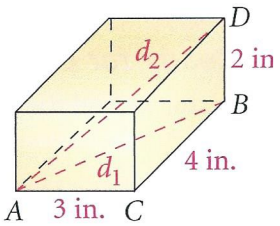


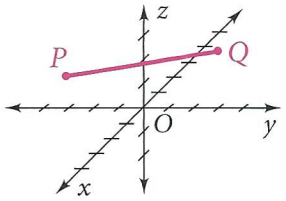
51. Geometry in 3 Dimensions The box at the right is a rectangular solid.



- **a.** Use $\triangle ABC$ to find the length of the diagonal of the base, d_1.
- **b.** Use $\triangle ABD$ to find the length of the diagonal of the box, d_2.
- **c.** You can **generalize** steps in parts (a) and (b).Use the fact that $AC^2 + BC^2 = {d_1}^2$ and ${d_1}^2 + BD^2 = {d_2}^2$ to write a one-step formula to find d_2.
- **d. Calculator** Use the formula you wrote to find the length of the longest fishing pole you can pack in a box with dimensions 18 in., 24 in., and 16 in.

Geometry in 3 Dimensions **Points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are points in a three-dimensional coordinate system. Use the following formula to find PQ. Leave your answer in simplest radical form.**

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



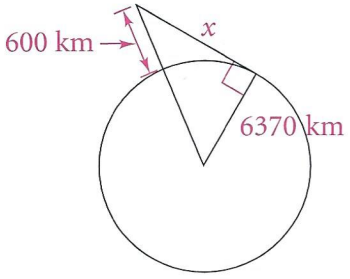
52. $P(0, 0, 0)$, $Q(1, 2, 3)$

53. $P(0, 0, 0)$, $Q(-3, 4, -6)$

54. $P(-1, 3, 5)$, $Q(2, 1, 7)$

55. $P(3, -4, 8)$, $Q(-1, 6, 2)$

56. Space The Hubble Space Telescope is orbiting Earth 600 km above Earth's surface. Earth's radius is about 6370 km. Use the Pythagorean Theorem to find the distance, x, from the telescope to Earth's horizon. Round your answer to the nearest ten kilometers.



not drawn to scale

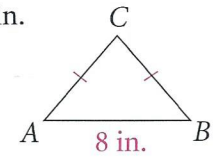
57. a. The ancient Greek philosopher Plato used the expressions $2n$, $n^2 - 1$, and $n^2 + 1$ to produce Pythagorean triples. Choose any integer greater than 1. Substitute for n and evaluate the three expressions.

b. Verify that your answers to part (a) form a Pythagorean triple.

58. Standardized Test Prep $\triangle ABC$ has perimeter 20 in. What is its area?

A. 12 in.^{2} **B.** 16 in.^{2} **C.** 24 in.^{2}

D. $8\sqrt{5}$ in.^{2} **E.** $16\sqrt{5}$ in.^{2}



Exercises MIXED REVIEW

Sketch each figure after a counterclockwise rotation of 90° about C.

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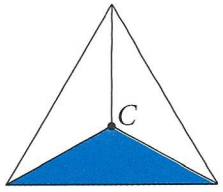

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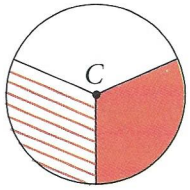

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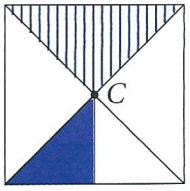

62. An angle is 87°. What is the measure of its complement?

Getting Ready for Lesson 5-4

Use a protractor to find the measures of the angles of each triangle.

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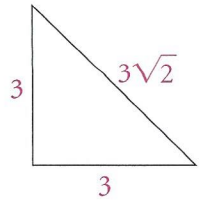

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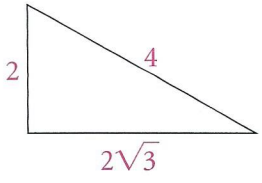

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